SOME NONLINEAR FUNCTIONS OF BERNOULLI AND EULER UMBRÆ

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Abstract. In a recent paper [5], Yi-Ping Yu has given some interesting nonlinear moments of the Bernoulli umbra; the aim of this paper is to show the probabilistic counterpart of these results and to extend them to Bernoulli polynomials.

1. Introduction

In a recent rich contribution, Yi-Ping Yu gives several nonlinear moments of the Bernoulli umbra $\mathfrak{B}$ defined by its generating function

$$\exp(z\mathfrak{B}) = \frac{z}{\exp(z)-1},\ |z| < 2\pi.$$

This umbra is related to the Bernoulli numbers as

$$\mathfrak{B}^n = B_n;$$

for example

$$B_0 = 1;\ B_1 = -\frac{1}{2};\ B_2 = \frac{1}{6};\ B_3 = 0;\ B_4 = -\frac{1}{30}.$$

and all odd orders Bernoulli numbers except B_1 equal 0.

Similarly, the Euler umbra $\mathfrak{E}$ is defined by the generating function

$$\exp(z\mathfrak{E}) = \operatorname{sech}(z)$$

We generalize here these umbræ and define the Bernoulli umbra $\mathfrak{B}(x)$ as

$$\exp(z\mathfrak{B}(x)) = \frac{ze^{zx}}{e^z-1} \tag{1.1}$$

and the Euler umbra $\mathfrak{E}(x)$ as

$$\exp(z\mathfrak{E}(x)) = \frac{2e^{zx}}{e^z+1}.$$

As a result,

$$\mathfrak{B}^n(x) = B_n(x)$$

and

$$\mathfrak{E}^n(x) = E_n(x),$$

respectively the Bernoulli and Euler polynomials of degree n.

The aim of this paper is to compute some nonlinear functions of these umbræ as probabilistic nonlinear moments. In the following, we denote the expectation operator

$$Eh(X) = \int h(x) f_X(x)\,dx$$

where f_X is the probability density function of the random variable X. We will use the following characterization of the Bernoulli and Euler umbræ.

Theorem 1. *The Bernoulli umbra $\mathfrak{B}(x)$ satisfies, for all admissible function h,*

$$h(\mathfrak{B}(x)) = Eh\left(x - \frac{1}{2} + \imath L_B\right)$$

where the random variable L_B follows a logistic distribution, with density

$$f_{L_B}(x) = \frac{\pi}{2} sech^2(\pi x), \; x \in \mathbb{R}.$$

Accordingly, the Euler umbra $\mathfrak{E}(x)$ satisfies, for all admissible function h,

$$h(\mathfrak{E}(x)) = Eh\left(x - \frac{1}{2} + \imath L_E\right)$$

where the random variable L_E follows the hyperbolic secant distribution

$$f_{L_E}(x) = sech(\pi x).$$

Proof. Since

$$\exp(\imath t \mathfrak{B}(x)) = E \exp\left(it\left(x - \frac{1}{2} + \imath L_B\right)\right),$$

by identification with (1.1), the random variable L_B has characteristic function

$$E\left(e^{\imath t L_B}\right) = \frac{\frac{t}{2}}{\sinh\left(\frac{t}{2}\right)}.$$

But from [6, 1.9.2]

$$\int_0^{+\infty} \operatorname{sech}^2(ax)\cos(xt)\,dx = \frac{\pi t}{2a^2}\operatorname{csch}\left(\frac{\pi t}{2a}\right)$$

so that, with $a = \pi$, the density of L_B is

$$f_{L_B}(x) = \frac{\pi}{2}\operatorname{sech}^2(\pi x),$$

which is a logistic density.

Accordingly, the characteristic function of the random variable L_E is

$$Ee^{\imath L_E t} = \operatorname{sech}\left(\frac{t}{2}\right).$$

From [6, 1.9.1],

$$\int_0^{+\infty} \operatorname{sech}(ax)\cos(xt)\,dx = \frac{\pi}{2a}\operatorname{sech}\left(\frac{\pi}{2a}t\right)$$

so that, with $a = \pi$, the density of L_E is

$$f_{L_E}(x) = \operatorname{sech}(\pi x).$$

Thus πL_0 follows an hyperbolic secant distribution. □

As a consequence, the Bernoulli polynomials read

$$B_n(x) = \mathfrak{B}(x)^n = E\left(x - \frac{1}{2} + \imath L_B\right)^n \tag{1.2}$$

and the Bernoulli numbers

$$B_n \quad = \mathfrak{B}^n = \mathfrak{B}(0)^n = \quad E\left(-\frac{1}{2} + \imath L_B\right)^n, \; n \geq 0.$$

Similarly, the Euler polynomials read

$$E_n(x) = \mathfrak{E}(x)^n = E\left(x - \frac{1}{2} + \imath L_E\right)^n$$

and the Euler numbers

$$E_n = 2^n \mathfrak{E}\left(\frac{1}{2}\right)^n = 2^n E(\imath L_E)^n.$$

We note from [7, p. 471] that the random variable L_B can also be obtained as

$$L_B = \frac{1}{2\pi}\log\frac{U}{1-U} = \frac{1}{2\pi}\log\frac{E_1}{E_2}$$

where U is uniformly distributed on $[-1,+1]$, E_1 and E_2 are independent with exponential distribution $f_E(x)=\exp(-x)\,,\ x\in[0,+\infty[$ and equality is in the sense of distributions. As for the random variable L_E, from [7], it can be obtained as

$$L_0=\frac{1}{\pi}\log|C|=\frac{1}{\pi}\left(\log|N_1|-\log|N_2|\right) \tag{1.3}$$

where C is Cauchy distributed and N_1 and N_2 are two independent standard Gaussian random variables.

2. THE MOMENT $\log\mathfrak{B}(x)$

We compute

$$\log\mathfrak{B}(x)=E\log\left(x-\frac{1}{2}+\imath L_B\right)$$

which, by symmetry, is equal to

$$\frac{1}{2}E\log\left(\left(x-\frac{1}{2}\right)^2+L_B^2\right)=\log\left|x-\frac{1}{2}\right|+\frac{1}{2}E\log\left(1+\frac{L_B^2}{\left(x-\frac{1}{2}\right)^2}\right)$$

but from [3, 2.6.30.2]

$$\int_0^{+\infty}\frac{\log\left(1+bz^2\right)}{\sinh^2 cz}dz\overset{d}{=}h(b,c)=\frac{2}{c}\left(\log\frac{c}{\pi\sqrt{b}}-\psi\left(\frac{c}{\pi\sqrt{b}}\right)\right).$$

Thus, by bisection of the angle $2\pi z$,

$$\int_0^{+\infty}\frac{\log\left(1+bz^2\right)}{\sinh^2 2\pi z}dz=\frac{1}{4}\int_0^{+\infty}\frac{\log\left(1+bz^2\right)}{\sinh^2\pi z\cosh^2\pi z}dz=\frac{1}{4}\int_0^{+\infty}\frac{\log\left(1+bz^2\right)}{\cosh^2\pi z}\left(\frac{\cosh^2\pi z}{\sinh^2\pi z}-1\right)dz$$

so that

$$\frac{\pi}{2}\int_{-\infty}^{+\infty}\frac{\log\left(1+bz^2\right)}{\cosh^2\pi z}dz \quad = \quad \pi\left(h(b,\pi)-4h(b,2\pi)\right).$$

We deduce, with $b=\left(x-\frac{1}{2}\right)^{-2}$,

$$\begin{aligned}\frac{1}{2}E\log\left(1+\frac{L_B^2}{\left(x-\frac{1}{2}\right)^2}\right) &= \frac{\pi}{2}\left(h(b,\pi)-4h(b,2\pi)\right)=\frac{\pi}{2}\left(\frac{2}{\pi}\left(\log\frac{1}{\sqrt{b}}-\psi\left(\frac{1}{\sqrt{b}}\right)\right)-\frac{4}{\pi}\left(\log\frac{2}{\sqrt{b}}-\psi\left(\frac{2}{\sqrt{b}}\right)\right)\right)\\ &= \log\frac{1}{\sqrt{b}}-2\log\frac{2}{\sqrt{b}}-\psi\left(\frac{1}{\sqrt{b}}\right)+2\psi\left(\frac{2}{\sqrt{b}}\right)\end{aligned}$$

and, using the identity

$$\psi(2z)=\frac{1}{2}\psi(z)+\frac{1}{2}\psi\left(z+\frac{1}{2}\right)+\log 2,$$

we obtain after simplification

$$E\log\left(\imath L_B+x-\frac{1}{2}\right)=\log\frac{1}{\sqrt{b}}+E\log\left(1+bL_B^2\right)=\psi\left(\frac{1}{2}+|x-\frac{1}{2}|\right).$$

For $x=1$, we recover the result by Y.-P. Yu, namely

$$E\log\left(\frac{1}{2}+\imath L_B\right)=\psi(1)=-\gamma.$$

3. THE MOMENT $\log \mathfrak{E}(x)$

This moment can be obtained according to the same approach, namely, again with $b = \left(x - \frac{1}{2}\right)^{-2}$,

$$\log \mathfrak{E}(x) = \log \frac{1}{\sqrt{b}} + \frac{1}{2} E \log\left(1 + bL_E^2\right)$$

where the latter expectation is now computed using [3, 2.6.30.1] as

$$\int_0^{+\infty} \frac{\log\left(1+bz^2\right)}{\cosh(\pi z)} dz = 2 \log \frac{\Gamma\left(\frac{3}{4} + \frac{1}{2\sqrt{b}}\right)}{\Gamma\left(\frac{1}{4} + \frac{1}{2\sqrt{b}}\right)} - \log \frac{1}{2\sqrt{b}}$$

so that

$$\log \mathfrak{E}(x) = \log 2 \frac{\Gamma^2\left(\frac{3}{4} + \frac{1}{2}|x - \frac{1}{2}|\right)}{\Gamma^2\left(\frac{1}{4} + \frac{1}{2}|x - \frac{1}{2}|\right)}.$$

4. THE MOMENTS $\mathfrak{B}^{-k}(x)$ AND $\mathfrak{E}^{-k}(x)$

By derivation of the preceding results, we deduce

$$\mathfrak{B}^{-1}(x) = E\left(x - \frac{1}{2} + \imath L_B\right)^{-1} = \frac{d}{dx} \log \mathfrak{B}(x)$$

so that we have

$$\mathfrak{B}^{-1}(x) = \begin{cases} \psi'(x), & x > \frac{1}{2} \\ -\psi'(-x+1), & x < \frac{1}{2} \\ 0 & x = \frac{1}{2} \end{cases}$$

and we remark that $\mathfrak{B}^{-1}(x)$ is not continuous in $x = \frac{1}{2}$. Since moreover for any integer $k \geq 1$

$$E\left(x - \frac{1}{2} + \imath L_B\right)^{-k} = \frac{(-1)^{k-1}}{(k-1)!} \frac{d^{k-1}}{dx^{k-1}} E\left(x - \frac{1}{2} + \imath L_B\right)^{-1}$$

we deduce

$$\mathfrak{B}^{-k}(x) = E\left(x - \frac{1}{2} + \imath L_B\right)^{-k} = \begin{cases} \frac{(-1)^{k-1}}{(k-1)!} \psi^{(k)}(x), & x > \frac{1}{2} \\ -\frac{1}{(k-1)!} \psi^{(k)}(-x+1) & x < \frac{1}{2} \end{cases}$$

and in a particular case $x = 1$, since $\psi^{(k)}(1) = (-1)^{k+1} k! \zeta(k+1)$,

$$\mathfrak{B}^{-k}(1) = E\left(\frac{1}{2} + \imath L_B\right)^{-k} = k.\zeta(k+1).$$

In the Euler case, we have

$$\mathfrak{E}^{-1}(x) = \frac{d}{dx} \log \mathfrak{E}(x) = \begin{cases} \psi\left(\frac{x+1}{2}\right) - \psi\left(\frac{x}{2}\right) & x > \frac{1}{2} \\ \psi\left(\frac{1-x}{2}\right) - \psi\left(1 - \frac{x}{2}\right) & x < \frac{1}{2} \\ 0 & x = \frac{1}{2} \end{cases}$$

and $\mathfrak{E}^{-1}(x)$ is not continuous in $x = \frac{1}{2}$.

More generally, for any integer $k \geq 1$,

$$\mathfrak{E}^{-k}(x) = \begin{cases} \frac{\left(-\frac{1}{2}\right)^{k-1}}{(k-1)!} \left(\psi^{(k-1)}\left(\frac{x+1}{2}\right) - \psi^{(k-1)}\left(\frac{x}{2}\right)\right), & x > -\frac{1}{2} \\ \frac{\left(\frac{1}{2}\right)^{k-1}}{(k-1)!} \left(\psi^{(k-1)}\left(\frac{1-x}{2}\right) - \psi^{(k-1)}\left(1 - \frac{x}{2}\right)\right), & x < -\frac{1}{2} \end{cases}$$

5. THE MOMENT $\log\sin\frac{\pi\mathfrak{B}}{2}$

This moment can be easily computed from the moment representation as follows

$$\begin{aligned}\log\sin\frac{\pi\mathfrak{B}}{2} &= E\log\sin\frac{\pi}{2}\left(-\frac{1}{2}+\imath L_B\right)=E\log\sin\left(-\frac{\pi}{4}-\imath\frac{\pi L_B}{2}\right)=E\log\sin\left(-\frac{\pi}{4}+\imath\frac{\pi L_B}{2}\right)\\ &= \frac{1}{2}E\log\sin\left(-\frac{\pi}{4}-\imath\frac{\pi L_B}{2}\right)\sin\left(-\frac{\pi}{4}+\imath\frac{\pi L_B}{2}\right)\end{aligned}$$

and expanding the product of sines we obtain

$$\frac{1}{2}E\log\left(\frac{1}{2}\cos\left(-\frac{\pi}{2}\right)+\frac{1}{2}\cos\left(\imath\pi L_B\right)\right)=-\frac{1}{2}\log 2+\frac{1}{2}E\log\cosh\left(\pi L_B\right).$$

But since

$$\pi L_B=\frac{1}{2}\log\frac{U}{1-U},$$

we deduce

$$\cosh\left(\pi L_B\right)=\frac{1}{2\sqrt{U\left(1-U\right)}}$$

so that, with $E\log U=-1$, we deduce

$$E\log\cosh\left(\pi L_B\right)=-\log 2+1$$

and the result

$$\log\sin\frac{\pi\mathfrak{B}}{2}=\frac{1}{2}-\log 2$$

follows.

6. THE POCHHAMMER $\left(\mathfrak{B}\left(x\right)\right)_n$

The Pochhammer symbol

$$\left(\mathfrak{B}+1\right)_n=\frac{\Gamma\left(\mathfrak{B}+n+1\right)}{\Gamma\left(\mathfrak{B}+1\right)}$$

has been evaluated in [1, p.149] as

$$\left(\mathfrak{B}+1\right)_n=\frac{n!}{\left(n+1\right)}.$$

We use the "intuitive argument" suggested by Carlitz [2] to compute its polynomial version $\left(\mathfrak{B}\left(x\right)\right)_n$ as follows: a generating function of $\left(\mathfrak{B}\left(x\right)\right)_n$ is

$$\begin{aligned}\varphi\left(x,t\right) &= \sum_{n=0}^{+\infty}\left(\mathfrak{B}\left(x\right)\right)_n\frac{t^n}{n!}=E\exp\left(-\left(x-\frac{1}{2}+\imath L_B\right)\log\left(1-t\right)\right)\\ &= \left(1-t\right)^{-\left(x-\frac{1}{2}\right)}E\exp\left(-\imath L_B\log\left(1-t\right)\right)\end{aligned}$$

with the characteristic function for the logistic density

$$E\exp\left(iL_B u\right)=\frac{\frac{u}{2}}{\sinh\left(\frac{u}{2}\right)}$$

so that

$$\varphi\left(x,t\right)=\left(1-t\right)^{-\left(x-\frac{1}{2}\right)}\frac{\frac{1}{2}\log\left(1-t\right)}{\sinh\left(\frac{\log(1-t)}{2}\right)}=-\left(1-t\right)^{-\left(x-1\right)}\frac{\log\left(1-t\right)}{t}$$

This term is identified as the derivative

$$\frac{d}{dx}\frac{\left(1-t\right)^{-\left(x-1\right)}}{t}=\frac{d}{dx}\sum_{n=0}^{+\infty}\frac{t^{n-1}}{n!}\left(x-1\right)_n=\sum_{n=0}^{+\infty}\frac{t^{n-1}}{n!}\frac{d}{dx}\left(x-1\right)_n$$

with

$$\frac{d}{dx}\left(x-1\right)_n=\left(x-1\right)_n\left(\psi\left(x+n-1\right)-\psi\left(x-1\right)\right)$$

so that the coefficient of $\frac{t^n}{n!}$ in $\varphi(x,t)$ is

$$(\mathfrak{B}(x))_n = \frac{(x-1)_{n+1}}{n+1}\left(\psi(x+n) - \psi(x-1)\right).$$

We recover the result by Nörlund by taking the limit case $x \to 1$ which is $\frac{n!}{n+1}$.

7. THE POCHHAMMER $(\mathfrak{E}(x))_n$

We use the same approach to compute the Pochhammer symbol of the Euler polynomial umbra; the generating function reads

$$\begin{aligned}\varphi(x,t) &= \sum_{n=0}^{+\infty} (\mathfrak{E}(x))_n \frac{t^n}{n!} = E\exp\left(-\left(x - \frac{1}{2} + \imath L_E\right)\log(1-t)\right) \\ &= (1-t)^{-\left(x-\frac{1}{2}\right)} E\exp\left(-\imath L_E \log(1-t)\right)\end{aligned}$$

with the characteristic function of the hyperbolic secant distribution

$$Ee^{\imath L_E t} = \operatorname{sech}\left(\frac{t}{2}\right)$$

so that

$$E\exp\left(\imath L_E \log(1-t)\right) = \operatorname{sech}\left(\frac{1}{2}\log(1-t)\right) = \frac{\sqrt{1-t}}{1-\frac{t}{2}}$$

and

$$\varphi(x,t) = \frac{1}{(1-t)^{x-1}\left(1-\frac{t}{2}\right)} = \sum_{n=0}^{+\infty} \frac{t^n}{n!}\frac{n!}{2^n}\sum_{k=0}^{n}\frac{(x-1)_k}{k!}2^k$$

so that

$$(\mathfrak{E}(x))_n = \frac{n!}{2^n}\sum_{k=0}^{n}\frac{(x-1)_k}{k!}2^k.$$

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